

The Natural Line-Breadth in Stochastic Electrodynamics

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Abstract

The fact that the formalism of stochastic electrodynamics includes the reaction force, makes the calculation of the natural line-breadth straightforward.

The evaluation of the relativistic correction to energy levels is greatly simplified in stochastic electrodynamics and brings out the concept of the propagation of the Coulomb field with the velocity of electromagnetic waves.

1. *Introduction*

Using strictly classical arguments and the concept of the universal zero-point field, an equation similar to Schrödinger's equation was derived (Surdin, 1971a).

The zero-point field is a random electromagnetic field at the absolute zero of temperature whose spectral density, for a one-dimensional case, is

$$\varepsilon(\omega) = \frac{K\omega^3}{3\pi c^3} \quad (1.1)$$

where K is a constant, having the dimension of action.

The above-mentioned derivation pertains to a linear, non-relativistic, motion of a particle of mass m , charge e , acted upon by an external field of force $F(x, t)$ and the universal zero-point field.

The corresponding equation of motion, a generalised Langevin equation, is

$$\ddot{x} + \beta\dot{x} - \frac{1}{m}F(x, t) = A(t) \quad (1.2)$$

β is the damping coefficient and $A(t)$ represents the zero-point field, considered to be a Markoff process.

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The wave function Ψ , representing the motion described by equation (1.2), satisfies the following modified Schrödinger equation

$$\frac{\partial \Psi}{\partial t} - \frac{iK}{2m} \cdot \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{K} [E - V(x, t)] \Psi = -\beta \Psi - \frac{Sm^2}{K^2} I_{\Psi}(x, t) \quad (1.3)$$

$V(x, t)$ is the potential function of the external field of force $F(x, t)$

$$-\frac{\partial V(x, t)}{\partial x} = F(x, t) \quad (1.4)$$

E is the energy of the system, S is the average spectral density of the energy absorbed by the particle from the zero-point field,

$$S = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\langle \left\langle \int_i^{i+\Delta t} A(t) dt \right\rangle^2 \right\rangle \quad (1.5)$$

$I_{\Psi}(x, t)$ is a function such that

$$\frac{\partial^2 I_{\Psi}(x, t)}{\partial x^2} = \Psi(x, t) \quad (1.6)$$

Apart from the terms of the second member, equation (1.3) is Schrödinger's equation, provided one has $K = \hbar$. According to stochastic electrodynamics K is a 'classical' quantity, the numerical value of which was evaluated independently of quantum mechanics. The calculated value was found to be, within a fraction of 1%, the same as that of \hbar (Surdin, 1971b).

The equation of motion of a linear, non-relativistic, harmonic oscillator, in which the damping force $2e^2 \bar{x}/3c^3$ is accounted for, may be reduced to equation (1.2), then $\beta = (2e^2/3mc^3)\omega_0^2 = \tau\omega_0^2$; ω_0 is the resonant frequency of the oscillator.

The application of equation (1.3) in this case has shown that the energy represented by the terms of the second member is small when compared to the ground state average energy and, to the first approximation, the right-hand side terms cancel each other (Surdin, 1971a).

It is the purpose of this paper to show that the second member, when fully considered and not simply in its approximate form, represents the natural line-breadth.

2. The Natural Line-Breadth of the Ground State of the Hydrogen Atom

For the study of the natural line-breadth, the pertinent wave function $\Psi(x)$ is time-independent; equation (1.3) becomes then

$$\frac{d^2 \Psi}{dx^2} + \frac{2m}{K^2} [E - V(x)] \Psi = -i \frac{2m}{K} \beta \Psi - i \frac{2Sm^3}{K^3} I_{\Psi}(x) \quad (2.1)$$

In fact, equation (2.1) is an integro-differential equation, since

$$I_{\Psi}(x) = \int dx \int \Psi(t) dt + f(x) \tag{2.2}$$

where $f(x)$ is a function of x depending on the initial conditions.

Equation (2.1) may be transformed into an integral equation, the formal solution of which is known. Such an approach is presented in the Appendix.

However, for the calculation of the line-breadth of the ground state of the hydrogen atom, a much simpler procedure will be followed.

It is well known that the necessary information about the ground state of the hydrogen atom is obtained for a Coulomb potential $V(r) = -e^2/r$ and for a centrally symmetrical wave function $\Psi(r)$. The Schrödinger equation in this case is

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right] \Psi + \frac{2m}{K^2} \left[E + \frac{e^2}{r} \right] \Psi = 0 \tag{2.3}$$

The solution of equation (2.3) is

$$\Psi = e^{-r/r_0}, \quad r_0 = \frac{K^2}{e^2 m} \quad \text{and} \quad E_0 = -\frac{K^2}{2m r_0^2} \tag{2.4}$$

The evaluation of the corresponding β and S , at the present state of our knowledge, is possible if one considers the equation of motion of the equivalent linear harmonic oscillator. The equation

$$\ddot{\mathbf{r}} + \tau_0 \omega_0^2 \dot{\mathbf{r}} + \frac{e^2}{m r_0^3} \mathbf{r} = \mathbf{A}(t) \tag{2.5}$$

where $\omega_0^2 = e^2/mr_0^3$ and $\omega_0 = K/mr_0^2$, corresponds to the case in hand, provided one bears in mind that the average ground state energy is negative.

The related modified Schrödinger equation is

$$\nabla^2 \Psi + \frac{2m}{K^2} \left[(E_0 + \Delta E) + \frac{e^2}{r} \right] \Psi = -i \frac{2m}{K} \beta \dot{\Psi} - \frac{i2Sm^3}{K^3} I_{\Psi}(r) \tag{2.6}$$

where the expressions (2.4) conserve their validity. To account for the terms on the right-hand side of equation (2.6) the energy E_0 was replaced by $E_0 + \Delta E$.

If one chooses the proper initial conditions for $I_{\Psi}(r)$, equation (2.6) yields

$$\Delta E = -iK\beta - \frac{iSm^2}{K} r_0^2 \tag{2.7}$$

As

$$\beta = \frac{2e^2}{3mc^3} \cdot \omega_0^2, \quad S = \frac{2e^2 K}{3m^2 c^3} \omega_0^3$$

and using the relations (2.4), equation (2.7) becomes

$$\Delta E = -i\frac{8}{3}\left(\frac{e^2}{Kc}\right)^3 |E_0| = -i\frac{8}{3}\alpha^3 |E_0| \quad (2.8)$$

ΔE is the line-breadth of the ground state of the hydrogen atom and α is the fine-structure constant.

The comparison of the natural line-breadth with the relativistic correction to energy levels appeared to present some interest.

3. The Relativistic Correction to Energy Levels

In quantum mechanics the relativistic correction to the energy levels of the hydrogen atom, due to the relativistic variation of the mass of the electron with velocity, is obtained by the application of the perturbation

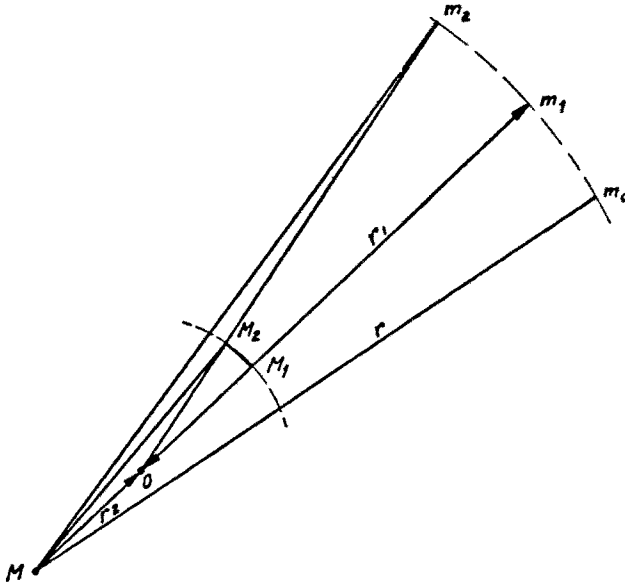


Figure 1.

theory (Bethe & Salpeter, 1957). Attention was drawn by Barut & Baiquni (1969) and Lieber (1972) to the delicate points in the application to this case of the perturbation theory.

In stochastic electrodynamics the approach is somewhat simpler. The electron, while in motion on its orbit around the proton, emits electromagnetic waves.† These waves probe the space surrounding the electron,

† The loss of energy, thus experienced, is exactly compensated by the gain of energy from the zero-point field (Surdin, 1973).

locate the proton, and inform the electron what the distance electron–proton is. This is equivalent of considering that the Coulomb potential propagates with the velocity of electromagnetic waves.

Consider the ground state of the hydrogen atom† whereby, for the sake of simplicity, the electron and the proton describe circular orbits around their centre, of mass. The situation is depicted in Fig. 1.

Let r_1 and r_2 be the radii of the circles described around the centre of mass O , respectively, by the electron of mass m and the proton of mass M . Then

$$r_0 = r_1 + r_2 \quad \text{and} \quad r_1 = \frac{r_0 M}{M + m}, \quad r_2 = \frac{r_0 m}{M + m} \quad (3.1)$$

The velocities v_1 of the electron and v_2 of the proton on their orbits are related by

$$v_2/v_1 = r_2/r_1 = m/M \quad (3.2)$$

While in motion on its orbit the electron emits an electromagnetic wave. The wave emitted at the time t_0 , when the electron is at m_0 , travels with velocity c , reaches M , then travels back and reaches the electron at m_2 . Thus the distance $r = \overline{m_0 M} = \overline{m_2 M}$, travelled by the electromagnetic wave between the electron and the proton, is slightly shorter than the distance r_0 . Hence, the energy of the ground state E_r will be slightly larger, in absolute value, than E_0 ; say

$$E_r = E_0 + \Delta E_r \quad (3.3)$$

One may evaluate ΔE_r in the following way: The distance r is approximated by

$$r = \overline{M M_2} + \overline{M_2 m_2} = [(2r_2)^2 - \overline{M_1 M_2^2}]^{1/2} + r_1 - r_2 \quad (3.4)$$

The time of propagation of the electromagnetic wave between m_0 and M is $t = r_0/c$, then

$$\overline{M_1 M_2} = \frac{v_2 r_0}{c} = \frac{m r_0 v_1}{M c}$$

Hence

$$r = \frac{2m r_0}{M + m} \left[1 - \frac{1}{4} \left(\frac{M + m}{M} \right)^2 \frac{v_1^2}{c^2} \right]^{1/2} + \frac{M - m}{M + m} r_0 \quad (3.5)$$

Neglecting terms of higher order in v_1^2/c^2 , one obtains from equation (3.5)

$$r = r_0 \left[1 - \frac{1}{4} \frac{m(M + m)}{M^2} \frac{v_1^2}{c^2} \right] \quad (3.6)$$

† The final results are easily extended to cover excited energy states.

The energy of the ground state before the application of the relativistic correction, $E_0 = -e^2/2r_0$, becomes after the correction is applied

$$E_r = E_0 + \Delta E_r = -e^2/2r \quad (3.7)$$

Hence

$$\Delta E_r = -\frac{1}{4} \frac{m(M+m)}{M} \frac{v_1^2}{c^2} |E_0| \quad (3.8)$$

Since $v_1^2/c^2 = \alpha^2$, one has, finally

$$\Delta E_r = -\frac{1}{4} \frac{m(M+m)}{M} \alpha^2 |E_0| \quad (3.9)$$

But for the factor $(M+m)/M$, this result is the same as the one obtained in quantum mechanics.

Comparison of ΔE_r with the natural line-breadth ΔE , shows that the latter is, approximately, two orders of magnitude larger than the relativistic correction. It should be borne in mind that the relativistic correction corresponds to the shifting of the uncorrected curve by ΔE_r .

4. Remarks

It is not surprising that the modified Schrödinger equation (1.3) or (2.1) allows one to calculate easily the natural line-breadth. This is due simply to the fact that the reaction force is included in this formalism.

In quantum mechanics, i.e. in the Schrödinger equation, the natural line-breadth is not included. To obtain the natural line-breadth one has to apply the perturbation theory† (Heitler, 1954).

In stochastic electrodynamics the relativistic correction to energy levels is obtained in a simple manner which introduces the notion of the propagation of Coulomb's potential with the velocity of electromagnetic waves. It is interesting to note here that a similar procedure was used to modify Newton's gravitational potential, in an attempt to account for the advance of the perihelion of Mercury and the bending of light rays by massive objects (Surdin, 1962).

Generalising, one may set forth the following principle: Two separate systems interact according to the information transferred from one system to the other.

This principle is included in stochastic electrodynamics, since the information is transferred by the all-pervading electromagnetic waves. For stationary motions the zero-point field compensates, on the average, the energy loss due to emission of electromagnetic waves by the moving system.

† Moreover, one has to assume that the probability of finding the atom in the excited state decreases exponentially with time.

In quantum mechanics, for instance in the case of the ground state of the hydrogen atom, the electron during its motion does not emit electromagnetic waves. No information about the proton is transferred to the electron, unless one considers the fluctuation of the vacuum, which is equivalent to the concept of the zero-point field.

Appendix

Equation (2.1) may be rewritten as

$$y''(x) = -\frac{2m}{K^2} [E + i\beta K] y(x) + \frac{2m}{K^2} V(x) y(x) - \frac{i2Sm^3}{K^3} \int_0^x \int_0^x y(t) dx dt + Ax + B \tag{A.1}$$

where Ψ was replaced by y , the terms $Ax + B$ represent the initial conditions on $I_\psi(x)$.

After reduction of the double integral, on the right-hand side, to a simple integral, viz:

$$\int_0^x \int_0^x y(t) dx dt = \int_0^x (x-t) y(t) dt \tag{A.2}$$

equation (A.1) becomes

$$y''(x) = ay(x) + bV(x)y(x) - c \int_0^x (x-t)y(t) dt + Ax + B \tag{A.3}$$

where

$$a = -\frac{2m}{K^2} (E + i\beta K), \quad b = \frac{2m}{K^2} \quad \text{and} \quad c = \frac{2iSm^3}{K^3}$$

Equation (A.3) is then integrated twice, at each step the reduction of the double integral into a simple integral is performed, finally one has

$$\left. \begin{aligned} y(x) &= a \int_0^x (x-t)y(t) dt + b \int_0^x (x-t)V(t)y(t) dt \\ &+ \frac{c}{6} \int_0^x (x-t)^3 y(t) dt + g(x) \end{aligned} \right\} \tag{A.4}$$

where

$$g(x) = \frac{A}{6} x^3 + \frac{B}{2} x^2 + Cx + D$$

A, B, C and D are constants, independent of x .

Let

$$N(x, t) = a(x-t) + b(x-t)V(t) - \frac{c}{6}(x-t)^3 \tag{A.5}$$

then equation (A.4) becomes

$$y(x) = g(x) + \int_0^x N(x, t) y(t) dt \quad (\text{A.6})$$

(A.6) is Volterra's integral of the second kind.

If, however, the modified Schrödinger equation is of the form (2.6), it may be rewritten as

$$\nabla^2 y(\mathbf{r}) = -\frac{2m}{K^2} [E + i\beta K] \cdot y(\mathbf{r}) + \frac{2m}{K^2} V(\mathbf{r}) y(\mathbf{r}) - \frac{i2Sm^3}{K^3} \int_a^{\mathbf{r}} \int_a^{\mathbf{r}} y(\mathbf{t}) d\mathbf{r} d\mathbf{t} \quad (\text{A.7})$$

the lower limits of integration are relative to the initial conditions. After reduction of the double integral into a simple integral, and conserving the notations of (A.3), one obtains

$$\nabla^2 y(\mathbf{r}) - ay(\mathbf{r}) = bV(\mathbf{r})y(\mathbf{r}) - c \int_a^{\mathbf{r}} (\mathbf{r} - \mathbf{t}) y(\mathbf{r}) d\mathbf{t} \quad (\text{A.8})$$

To solve equation (A.8) one may follow the procedure indicated by Arfken (1970), viz: let

$$\left. \begin{aligned} \Phi(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int y(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \\ y(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}} \int \Phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \end{aligned} \right\} \quad (\text{A.9})$$

where

$$\frac{\mathbf{p}}{K} = \mathbf{k} \quad (\text{A.10})$$

Rewrite equation (A.8) as

$$\begin{aligned} \int (\nabla^2 - a) y(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} &= b \int V(\mathbf{r}) y(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \\ &\quad - c \int \int_a^{\mathbf{r}} (\mathbf{r} - \mathbf{t}) y(\mathbf{t}) e^{-i\mathbf{k}\cdot\mathbf{t}} d\mathbf{r} d\mathbf{t} \end{aligned} \quad (\text{A.11})$$

then, combining (A.9) and (A.11), one obtains

$$\begin{aligned} -(2\pi)^{3/2} (k^2 + a) \Phi(\mathbf{k}) &= \frac{b}{(2\pi)^{3/2}} \int V(\mathbf{r}) \int \Phi(\mathbf{k}') e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} d\mathbf{r} d\mathbf{k}' \\ &\quad - c \int_a^{\mathbf{r}} (\mathbf{r} - \mathbf{t}) d\mathbf{r} d\mathbf{t} \int \Phi(\mathbf{k}') e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{t}} d\mathbf{k}' \end{aligned} \quad (\text{A.12})$$

Let

$$\left. \begin{aligned} G(\mathbf{k}, \mathbf{k}') &= \frac{b}{(2\pi)^3} \int V(\mathbf{r}) e^{-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}} d\mathbf{r} \\ F(\mathbf{k}, \mathbf{k}') &= \frac{c}{(2\pi)^3} \int_a^{\mathbf{r}} d\mathbf{r} \int_a^{\mathbf{r}} (\mathbf{r} - \mathbf{t}) e^{-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{t}} d\mathbf{t} \end{aligned} \right\} \quad (\text{A.13})$$

then

$$-(k^2 + a) \Phi(\mathbf{k}) = \int [G(\mathbf{k}, \mathbf{k}') - F(\mathbf{k}, \mathbf{k}')] \Phi(\mathbf{k}') d\mathbf{k}' \quad (\text{A.14})$$

Equation (A.14) is Volterra's integral equation of the first kind.

Once $\Phi(\mathbf{k})$ is determined, the second equation (A.9) gives $y(\mathbf{r})$.

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